JOURNAL OF APPROXIMATION THEORY 17, 344–347 (1976)

Best Approximation to Functions with Restricted Derivatives

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Communicated by E. W. Cheney

Received January 9, 1975

1. INTRODUCTION

Let $0 \leqslant k_1 < \cdots < k_p$ be fixed integers and let ℓ_i and μ_i, $i = 1,..., p$, be fixed extended real valued functions on $[-1, +1]$ which satisfy the following conditions:

(i) ℓ_i may take the value $-\infty$ but never $+\infty$;

(ii) μ_i may take the value $+\infty$ but never $-\infty$;

(iii) $X_i^- = \{x \in [-1, +1]; l_i(x) = -\infty\}$ and $X_i^+ = \{x \in [-1, +1]; \mu_i(x) = +\infty\}$ are open subsets of $[-1, +1]$; (1)

(iv) ℓ_i is continuous on the complement of X_i^- and μ_i is continuous on the complement of X_i^+;

(v) $\ell_i(x) < \mu_i(x)$ for all $x \in [-1, +1]$.

These conditions ensure that there is $\epsilon > 0$ for which $\mu_i(x) - \ell_i(x) \geqslant \epsilon$ for all x in $[-1, +1]$. Let f have k_p continuous derivatives on $[-1, +1]$ and assume that for $i = 1,..., p$ and x in $[-1, +1]$ we have

$$\ell_i(x) < f^{(k_i)}(x) < \mu_i(x). \tag{2}$$

It is easy to see that there is $\delta > 0$ so that for all x in $[-1, +1]$ and $i = 1,..., p$ we have

$$\min(\mu_i(x) - f^{(k_i)}(x), \qquad f^{(k_i)}(x) - \ell_i(x)) \geqslant \delta. \tag{3}$$

For each nonnegative integer n let H_n *denote the algebraic polynomials of degree n or less*, and let $\|\cdot\|$ *denote the uniform norm* on $[-1, +1]$.

Let f be continuous on $[-1, +1]$. For each integer $n = 0, 1, 2,...$ define $E_n(f) = \inf_{p_n \in H_n} \|f - p_n\|$. If $p \in H_n$ and $\|f - p\| = E_n(f)$ then p is called the *polynomial of best approximation to f from* H_n. It is well known that for each n p exists and is unique.

In this paper we will give new sufficient conditions on f to ensure that if $p_n \in H_n$ is the polynomial of best approximation to f then for n sufficiently large

$$\ell_i(x) < p_n^{(k_i)}(x) < \mu_i(x). \tag{4}$$

Roulier [3] studies this problem when we have either

$$\mu_i \equiv +\infty \qquad \text{and} \qquad \ell_i \equiv 0$$

or

$$\mu_i \equiv 0 \qquad \text{and} \qquad \ell_i \equiv +\infty.$$

Kimchi and Leviatan [1] also study this problem. In particular they show that (4) will hold if f has $2k_p$ continuous derivatives, if

$$\sum 1/n\ \omega(f^{(2k_p)}, 1/n^{1/2}) < +\infty, \tag{5}$$

and if f satisfies (2). ω is the modulus of continuity of $f^{(2k_p)}$.

The main theorem of this paper shows that (5) may be omitted entirely.

Malozemov [2] proved the following theorem.

THEOREM 1.1. *Let f have r continuous derivatives on $[-1, +1]$. Let $\delta_n(x) = (1/n)((1/n) + (1 - x^2)^{1/2})$. Then for each integer $n \geqslant r$ there is a polynomial $q_n \in H_n$ such that for each $k = 0, 1,..., r$ and each x in $[-1, +1]$*

$$|f^{(k)}(x) - q_n^{(k)}(x)| \leqslant C_r \delta_n(x)^{r-k}\, \omega(f^{(r)}, \delta_n(x)).$$

C_r is a constant depending only on r.

2. The Main Theorems

THEOREM 2.1. *Let f have r continuous derivatives on $[-1, +1]$ and let k be a positive integer which satisfies $2k \leqslant r$. For each n let $p_n \in H_n$ be the polynomial of best approximation to f on $[-1, +1]$. Then*

$$\lim_{n \to \infty} \| f^{(k)} - p_n^{(k)} \| = 0.$$

Proof. Let $\{q_n\}_{n=0}^{\infty}$ be a sequence of polynomials as in Theorem 1.1. Then

$$\| f^{(s)} - q_n^{(s)} \| \leqslant (C/n^{r-s})\, \omega(f^{(r)}, 1/n) \tag{6}$$

for $s = 0, 1,..., r$. For each $n = 0, 1,...$ let r_n be the nth degree polynomial of best approximation to $f - q_n$. Then $q_n + r_n$ is the nth degree polynomial of best approximation to f. Write $p_n = q_n + r_n$. Note that

$$E_n(f) \leqslant (K/n^r)\, \omega(f^{(r)}, 1/n). \tag{7}$$

Note also that

$$\begin{aligned} \| r_n \| &\leqslant \| f - p_n \| + \| f - q_n \| \\ &\leqslant (C_1/n^r)\, \omega(f^{(r)}, 1/n). \end{aligned}$$

Thus, by the Markov inequality we have

$$\| r_n^{(s)} \| \leqslant (C_1/n^{r-2s})\, \omega(f^{(r)}, 1/n).$$

Clearly if $2s \leqslant r$ we have

$$\lim_{n \to \infty} \| r_n^{(s)} \| = 0.$$

Hence, if $2s \leqslant r$ we have using this and (6) that

$$\begin{aligned} \| f^{(s)} - p_n^{(s)} \| &\leqslant \| f^{(s)} - q_n^{(s)} \| + \| r_n^{(s)} \| \\ &\leqslant (C/n^{r-s})\, \omega(f^{(r)}, 1/n) + \| r_n^{(s)} \|. \end{aligned}$$

Hence $\lim_{n\to\infty} \| f^{(s)} - p_n^{(s)} \| = 0$ if $2s \leqslant r$. This is the desired result. ∎

THEOREM 2.2. *Let $k_1 < k_2 < \cdots < k_p$ be fixed nonnegative integers as above and let ℓ_i and μ_i $i = 1, 2,\ldots, p$ be extended real-valued functions as above. Let $f^{(2k_p)}$ be continuous on $[-1, +1]$. Assume that for all x in $[-1, +1]$ we have for $i = 1,\ldots, p$*

$$\ell_i(x) < f^{(k_i)}(x) < \mu_i(x). \tag{8}$$

For $n = 0, 1, 2,\ldots$ let $p_n \in H_n$ be the polynomial of best approximation to f on $[-1, +1]$. Then for n sufficiently large we have

$$\ell_i(x) < p_n^{(k_i)}(x) < \mu_i(x) \qquad \text{for } -1 \leqslant x \leqslant 1.$$

Proof. This is a simple consequence of Theorem 2.1. ∎

The following is a simple corollary to Theorem 2.2 concerning monotone approximation.

COROLLARY. *Let f have two continuous derivatives on $[-1, +1]$ and assume $f'(x) \geqslant \delta > 0$ on $[-1, +1]$. Then for n sufficiently large the p in H_n of best approximation to f is increasing on $[-1, +1]$.*

3. Conclusions

These theorems improve on the results in [1, 3]. It is not known yet whether it is necessary for f to have $2k_p$ continuous derivatives for Theorem 2.2 to remain valid. In particular, is the above corollary true if we only assume f' is continuous?

References

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